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Local Strong Uniqueness

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A study of local strong uniqueness is given. Concepts of local strong uniqueness and directional local strong uniqueness of at best, at worst, or exact rate ϕ are introduced. The relation of local strong uniqueness to the "conditioning" of the approximation problem and to the modulus of convexity of the underlying space are noted. Special emphasis is given to L^p approximation. Of particular interest here is that a continuum of local strong uniqueness rates is possible for L^p, $1 < p < 2$; whereas, for $2 < p < \infty$, only one of two possible local strong uniqueness rates can occur for each approximation problem.

1. INTRODUCTION

The concept of strong uniqueness was first introduced by Newman and Shapiro [11]. This quantitative estimate for uniqueness, called the strong uniqueness theorem, was given in [11] for uniform approximation from a Haar subspace. In the case of real functions, for B a compact Hausdorff space, if $f \in C(B) \setminus V$ has $v^* \in V$ as its unique best approximation from V, then there exists $\gamma = \gamma(f) > 0$ such that

$$\|f - v\| \geqslant \|f - v^*\| + \gamma \|v - v^*\| \tag{1}$$

for all $v \in V$ with $\| \ \|$ denoting the uniform norm on $C(B)$. Furthermore, since by the triangle inequality one has that $\|f - v\| \leqslant \|f - v^*\| + \|v - v^*\|$ holds, it is clear that this estimate is "best possible" with respect to

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$\|v-v^*\|$. For complex-valued functions the corresponding result has the form

$$\|v-v^*\| \leqslant K_1\{\|f-v\|-\|f-v^*\|\}^{1/2}+K_2\{\|f-v\|-\|f-v^*\|\}, \quad (2)$$

where it was noted [11] that the term with exponent 1/2 is, of course, dominant in "all applications of interest," and the second term is required only to preserve the inequality when $\|v\|$ is large. This inequality can be viewed as a local strong uniqueness result since for each $M>0$ there exists $\gamma=\gamma(M,f)>0$ such that

$$\|f-v\| \geqslant \|f-v^*\|+\gamma\|v-v^*\|^2 \quad (3)$$

holds for all $v \in V$ satisfying $\|v-v^*\| \leqslant M$. In addition, this estimate is known by example to be sharp. More recently, H.-P. Blatt [2] has shown that in this setting, if B has at most n ($=\dim V$) isolated points then the strong uniqueness inequality (1) holds almost everywhere in $C(B)$. Thus in this setting, local strong uniqueness of order 2 (i.e., Eq. (3)) actually can occur although it is somewhat of an exceptional occurrence. A related open question here is whether local strong uniqueness of order α, $1<\alpha<2$, can also occur for specific examples (see next section for definitions).

More recently, Y. Fletcher and J. A. Roulier [10] and D. Schmidt [12] have shown that there exist examples of best uniform monotone approximation problems where local strong uniqueness of order 2 (i.e., (3)) holds and is sharp. As in [2], it is also the case here that local strong uniqueness of order 2 holding occurs as an exceptional case [5, Thm. 12]. Additional work concerning local strong uniqueness in constrained uniform approximation can be found in [4] and in [1, 9] for the L^p norms.

In what follows, a detailed study of local strong uniqueness will be given. We shall study local strong uniqueness properties at a fixed element not in the set of approoximants, rather than seeking one estimate for all elements which might be approximated. We begin by giving refined definitions of local strong uniqueness that distinguish the quantitative behaviour of uniqueness in terms of "at best," "at worst," or "exact" growth estimates. Next, a concept of directional local strong uniqueness is introduced. Various applications of these concepts are given. Included among these is that the modulus of convexity of the space in which we are approximating is a lower bound on the local strong uniqueness at each point. Also, a careful study of L^p shows that the results in the literature to date for local strong uniqueness are exact for $p \geqslant 2$, but not exact for $1<p<2$. In fact, a continuum of results is possible in this latter case. Finally, we wish to note that local strong uniqueness estimates, when available, are a true measure of the conditioning of the particular approximation problem under consideration.

2. Definitions and Main Concepts

Let X be a Banach space with norm $\| \ \|$. Let $V \subseteq X$ be an n-dimensional unicity subspace of X. Fix $f \in X$ and let $v^* \in V$ denote the unique best approximation to f from V. In this setting we define the following.

Definition 1. Let $\phi \in C[0, \infty)$ satisfy $\phi(0) = 0, \phi(t) \leqslant t$, with $\phi(t)$ strictly increasing. We say that local strong uniqueness of rate *at worst* ϕ holds for f at v^* with respect to V if corresponding to each $M > 0$ there exists $\gamma = \gamma(M, f) > 0$ such that

$$\|f - v\| \geqslant \|f - v^*\| + \gamma\phi(\|v - v^*\|) \tag{4}$$

for all $v \in V$ satisfying $\|v - v^*\| \leqslant M$.

Definition 2. With ϕ as in Definition 1, we say that local strong uniqueness of rate *at best* ϕ holds for f at v^* with respect to V provided there exist $\{v_k\}_{k=1}^{\infty}$ in V with $v_k \neq v^*$, $v_k \to v^*$, and $\beta = \beta(f) > 0$ such that

$$\|f - v_k\| \leqslant \|f - v^*\| + \beta\phi(\|v_k - v^*\|) \tag{5}$$

for all k.

Definition 3. Let ϕ be as in Definition 1. We say that local strong uniqueness of rate ϕ holds for f at v^* with respect to V provided that both local strong uniqueness of rate at worst ϕ and of rate at best ϕ hold for f at v^* with respect to V.

Note that in this terminology whenever local strong uniqueness holds it is necessarily sharp. This is in contrast to earlier definitions where Definition 2 was not required, but usually satisfied without additional comment. Further, since for any $v \in V$, $\|f - v\| \leqslant \|f - v^*\| + \|v - v^*\|$ by the triangle inequality the requirement that $\phi(t) \leqslant t$ hold is not a true restriction on the problem. That is, for any $\phi \in C[0, \infty)$ with $\phi(0) = 0$, ϕ strictly increasing, and $\lim_{x \to 0^+} \phi(t)/t = \infty$, local strong uniqueness of rate ϕ cannot hold as it would force $\|v - v^*\| \geqslant \gamma\phi(\|v - v^*\|)$ which is not possible.

If, in the above definitions, $\phi(t) = t^{\alpha}, \alpha \geqslant 1$, then one also says that local strong uniqueness of *order* at worst α, *order* at best α, or *order* α, respectively, holds in the three definitions. Observe that this is a slight change in terminology from some of the earlier papers where strong uniqueness of order $r, 0 \leqslant r \leqslant 1$, was used for $\phi(t) = t^{1/r}$ (i.e., our order α is their order $1/r$).

Refining the above definitions further, one can consider *directional* local strong uniqueness. This approach is useful for actual computations of the rate of local strong uniqueness in specific examples in that it reduces the problem to that of a real-valued function of a single variable. Thus for $M>0$ and $v \in V$ both fixed, with $\|v\|=1$ and ϕ as in the previous definitions, we have the following.

DEFINITION 4. We say that local strong uniqueness of rate *at worst* ϕ holds for f at v^* in the direction v provided there exits $\gamma=\gamma(M, f, v)>0$ such that

$$\|f-(v^*+\varepsilon v)\| \geqslant \|f-v^*\|+\gamma\phi(\varepsilon) \tag{6}$$

holds for all ε, $0 \leqslant \varepsilon \leqslant M$.

DEFINITION 5. We say that local strong uniqueness of rate *at best* ϕ holds for f at v^* in the direction v provided that there exists $\varepsilon_k \downarrow 0$, $\varepsilon_k \neq 0$, and $\beta=\beta(M, f, v)>0$ such that

$$\|f-(v^*+\varepsilon_k v)\| \leqslant \|f-v^*\|+\beta\phi(\varepsilon_k) \tag{7}$$

holds for all k.

DEFINITION 6. We say that local strong uniqueness of rate ϕ holds for f at v^* in the direction v provided both local strong uniqueness of rate at worst ϕ and of rate at best ϕ in the direction v holds for f at v^*.

EXAMPLE 1. In R^3 consider those points such that $\rho(x, y, z) = x^2+y^4+z^6 = 1$. This surface induces the norm $\|(x, y, z)\| = \inf\{\lambda: \rho(x/\lambda, y/\lambda, z/\lambda)<1, \lambda>0\}$. In the notation from above, for $V = \{(x, y, z): x=0\}$ and $f=(1, 0, 0)$ we have that $v^*=(0, 0, 0)$ and local strong uniqueness of order 4 holds in the direction $(0, 1, 0)$ while local strong uniqueness of order 6 holds in the direction $(0, 0, 1)$. Likewise, for $V=\{(x, y, z): y=0\}$ and $f=(0, 1, 0)$ we have that $v^*=(0, 0, 0)$ and local strong uniqueness of order 2 holds in the direction $(1, 0, 0)$ while local strong uniqueness order 6 holds again in the direction $(0, 0, 1)$. Furthermore, for $V=\{(x, y, z): y=z=0\}$ and $f=(0, 1, 0)$ we have that $v^*=(0, 0, 0)$ and local strong uniqueness of order 2 holds for f at v^* with respect to V.

We note that this definition is an extension of the concept of order of contact between surfaces [3]. Here the "surfaces" are the boundary of the ball centered at f of radius $\|f-v^*\|$ and the subspace V. Thus, that local strong uniqueness of rate t^2 holds in least-squares approximation is a restatement of the fact that a true sphere and a tangent plane have order of contact 2. Note also that the above concepts can be used to establish local strong uniqueness of rate ϕ for f at v^*. Indeed, if a positive constant γ can be shown to exist such that $\gamma(M, f, v) \geqslant \delta > 0$ for all $v \in V$ satisfying $\|v\| = 1$ then it clearly follows that local strong uniqueness of rate at worst ϕ holds for f at v^*. Furthermore, if there exists one v satisfying $\|v\| = 1$ for which local strong uniqueness of rate at best ϕ holds for f at v^* in the direction of v then local strong uniqueness of rate ϕ will hold for f at v^*. One way in which this can occur is stated in the following lemma.

LEMMA 1. *Suppose that local strong uniqueness of rate at worst ϕ holds for f at v^* in each direction v, $v \in S = \{v \in V: \|v\| = 1\}$, and further that there exists a continuous selection of the local strong uniqueness constant $\gamma(M, f, v)$ as a function of v. Then, local strong uniqueness of rate at worst ϕ holds for f at v^*.*

Proof. Since S is a closed bounded subset of a finite-dimensional subspace it is compact. Hence a continuous positive function defined on this set must have a positive minimum. ∎

As in the standard theory, local strong uniqueness results give local bounds for the best approximation operator. Specifically, the best approximation operator τ, defined by letting $\tau(f)$ be the unique best approximation to f from V, is a map from X to V and the local behaviour of this operator is bounded by ϕ^{-1} for any ϕ satisfying (4) for f at $\tau(f)$. Indeed, by the usual proof [6, p. 82] one has that

$$\begin{aligned}\gamma\phi(\|\tau(f) - \tau(g)\|) &\leqslant \|f - \tau(g)\| - \|f - \tau(f)\| \\ &\leqslant \|f - g\| + \|g - \tau(g)\| - \|f - \tau(f)\| \\ &\leqslant \|f - g\| + \|g - \tau(f)\| - \|f - \tau(f)\| \leqslant 2\,\|f - g\|\end{aligned}$$

for $\|g\| \leqslant 2\,\|f\|$ and $M = 6\,\|f\|$ say. In the case $\phi(t) = t^\alpha$, $\alpha \geqslant 1$, then this is a local Lipschitz condition of order $1/\alpha$.

Note also that if local strong uniqueness of rate ϕ holds for f at v^* then this gives one a conditioning measure of the best approximation problem for f from V relative to the distance function. That is, if $\operatorname{dist}(f, V) \leqslant \|f - v\| \leqslant \operatorname{dist}(f, V) + \varepsilon$ for some $v \in V$ then we must have that $\|v - v^*\| \leqslant \phi^{-1}(\varepsilon/\gamma)$. This estimate is a formal expression, for example, of the oft discovered "ill-conditioning" of the least-squares approximation problem. That is, for linear least-squares approximation one has that local strong uniqueness of order 2 holds. Thus a good approximation of f (say $O(\varepsilon)$) need only be a fair approximation of v^* (say $O(\sqrt{\varepsilon})$).

It is known that strong uniqueness is related to the rate of convergence of certain approximation schemes [7]. Some of these results are actually local in nature, and here it is local strong uniqueness that is important. For example, in the discussion of Newton's method in Cromme [7] the *local* equivalence of strong uniqueness properties of a nonlinear and a linear map are used to ensure convergence.

That the concepts of strong uniqueness, modulus of convexity, and modulus of smoothness are closely related in L^p spaces is well known [14, 8]. The following shows that these concepts remain related in a more general setting. For a Banach space X, let $S = \{x \in X: \|x\| = 1\}$. Then $\delta(\varepsilon)$, the *modulus of convexity* of X, is defined by

$$\delta(\varepsilon) = \inf\{1 - \|(x+y)/2\| : x, y \in S, \|x-y\| = \varepsilon\} \qquad \text{for} \quad 2 > \varepsilon > 0.$$

Suppose that V is a finite-dimensional subspace of X, $f \in X \backslash V$, and that v^* is the unique best approximation from V to f. Furthermore, suppose that local strong uniqueness of order ϕ in the direction v holds here. For simplicity, assume that $\|f - v^*\| = 1$. Define $\psi(\varepsilon) = \|f - (v^* + \varepsilon v)\| - \|f - v^*\|$. Letting $\varepsilon_k \downarrow 0$ be a sequence as in Definition 5 for ϕ, we have that $\psi(\varepsilon_k) \leqslant \beta\phi(\varepsilon_k) \forall k$. Normalizing $f - (v^* + \varepsilon v)$, one gets that $(f - (v^* + \varepsilon v))/(1 + \psi(\varepsilon))$ is the corresponding unit vector as $\|f - (v^* + \varepsilon v)\| = 1 + \psi(e)$. Now for each ε_k, we have that

$$\begin{aligned} &\|(f - v^*) - (f - (v^* + \varepsilon_k v))/(1 + \psi(\varepsilon_k))\| \\ &\quad = \|(f - v^*)\,\psi(\varepsilon_k) + \varepsilon_k v\|/(1 + \psi(\varepsilon_k)) \geqslant \frac{\varepsilon_k - \psi(\varepsilon_k)}{1 + \psi(\varepsilon_k)}. \end{aligned}$$

If we assume $\lim_{t \to 0^+} \phi(t)/t = 0$, then for all k sufficiently large, $(\varepsilon_k - \psi(\varepsilon_k)/(1 + \psi(\varepsilon_k)) \geqslant \rho\varepsilon_k$ where $0 < \rho < 1$ with ρ fixed. Thus for this ρ,

$$\begin{aligned} \delta(\rho\varepsilon_k) &\leqslant 1 - \frac{1}{2}\|(f - v^*) + (f - (v^* + \varepsilon_k v))/(1 + \psi(\varepsilon_k))\| \\ &= 1 - \frac{1}{2(1 + \psi(\varepsilon_k))}\|2(f - v^*) + \psi(\varepsilon_k)(f - v^*) - \varepsilon_k v\| \\ &= 1 - \frac{\|(f - v^*) + (\psi(\varepsilon_k)/2)(f - v^*) - (\varepsilon_k/2)v\|}{1 + \psi(\varepsilon_k)} \\ &= 1 - \frac{\|f - (v^* + (\varepsilon_k/2)v) + (\psi(\varepsilon_k)/2)(f - v^*)\|}{1 + \psi(\varepsilon_k)} \\ &\leqslant 1 - \frac{\|f - (v^* + (\varepsilon_k/2)v)\| - (\psi(\varepsilon_k)/2)}{1 + \psi(\varepsilon_k)} \\ &= 1 - \frac{1 + \psi(\varepsilon_k/2) - (\psi(\varepsilon_k)/2)}{1 + \psi(\varepsilon_k)} = 1 - \frac{1 + \psi(\varepsilon_k) + \psi(\varepsilon_k/2) - (3/2)\,\psi(\varepsilon_k)}{1 + \psi(\varepsilon_k)} \\ &= \frac{(3/2)\,\psi(\varepsilon_k) - \psi(\varepsilon_k/2)}{1 + \psi(\varepsilon_k)} \leqslant \frac{3}{2}\,\psi(\varepsilon_k). \end{aligned}$$

That is, for fixed ρ, $0<\rho<1$, and all k sufficiently large, $\delta(\rho\varepsilon_k)\leq\frac{3}{2}\beta\psi(\varepsilon_k)$ holds, showing that the modulus of convexity is essentially a lower bound for the rate of local strong uniqueness at any given point. However, this is generally a very pessimistic lower bound. Indeed, for Example 1, we saw that various local strong uniqueness orders held in various settings with the best being order 2 and the worst order 6. By considering the vectors $x=(\eta, 0, \varepsilon/2)$ and $y=(\eta, 0, -\varepsilon/2)$ where $\eta>0$ is chosen so that $\|x\|=\|y\|=1$ (i.e., $1/\eta^2+\varepsilon^6/2^6\eta^6=1$) it can easily be seen that the modulus of convexity, $\delta(\varepsilon)$, of this space satisfies $\delta(\varepsilon)\leq k\varepsilon^6$, where k is independent of ε. Thus, the direct connection between the concepts of modulus of convexity and local strong uniqueness is rather weak. In some sense the modulus of convexity measures the flattest spot on the unit ball for the full unit sphere, whereas local strong uniqueness is measuring the flatness of a restricted unit ball at a specific point.

3. Local Strong Uniqueness in L^p

In this section we extend the results of [1] and compare these results with some global estimates of Smarzewski [13, 14]. We begin by considering L^p for $p\geq 2$. Here we shall show that the local strong uniqueness results of [1] which are *at worst* estimates are also *at best* estimates, showing that their estimates are sharp. Thus, let $L^p=L^p(S, \Sigma, \mu)$, $1\leq p<\infty$, be the Banach space of all μ-measurable extended real-valued functions f on S with

$$\|f\|_p=\left(\int_S |f(t)|^p\,d\mu(t)\right)^{1/p}<\infty,$$

where (S, Σ, μ) is a finite positive measure space, and let V be an n-dimensional subspace of L^p. To obtain the desired result for $2\leq p<\infty$, we use the concept of directional local strong uniqueness.

Theorem 1. *For $2\leq p<\infty$, fix $f\in L^p$ and $M>0$. Let $v^*\in V$ be the unique best L^p approximation to f from V. The following hold:*

(i) *If for each nonzero $v\in V$, $\mu\{\operatorname{supp}(f-v^*)\cap\operatorname{supp}(v)\}\neq 0$ then local strong uniqueness of order 2 holds for f at v^*;*

(ii) *If there exist $\hat{v}\in V$, $\|\hat{v}\|_p=1$, with $\mu\{\operatorname{supp}(f-v^*)\cap\operatorname{supp}(\hat{v})\}=0$, then local strong uniqueness of order p holds for f at v^*.*

Proof. First, consider (i). From [1] we have that local strong uniqueness of order at worst 2 holds for f at v^*, i.e., there exists $\gamma = \gamma(f, M) > 0$ such that

$$\|f - v\|_p \geq \|f - v^*\|_p + \gamma \|v - v^*\|_p^2 \tag{8}$$

for all $v \in V$ satisfying $\|v\|_p \leq M$. This being the more difficult inequality, we need only demonstrate that local strong uniqueness of order at best 2 holds for f at v^*. To this end fix $v \in V$, $\|v\|_p = 1$, and define $F(\varepsilon) = \|f - v_\varepsilon\|_p$, $v_\varepsilon = v^* + \varepsilon v$, $-M \leq \varepsilon \leq M$. Note that

$$F'(\varepsilon) = 1/p\left(\int |f - v_\varepsilon|^p \, d\mu\right)^{1/p - 1}\left(p\int |f - v_\varepsilon|^{p-2}(f - v_\varepsilon)(-v)\, d\mu\right)$$

and that

$$F''(\varepsilon) = (1-p)\, \|f - v_\varepsilon\|_p^{1-2p}\left(\int |f - v_\varepsilon|^{p-2}(f - v_\varepsilon)(-v)\, d\mu\right)^2$$
$$+ \|f - v_\varepsilon\|^{1-p}\left((p-1)\int |f - v_\varepsilon|^{p-2} v^2 \, d\mu\right)$$

with $F'(0) = 0$. Thus, $F(\varepsilon) = F(0) + \frac{1}{2}F''(\eta)\varepsilon^2$ for some η between 0 and ε. Since $|F''(\eta)| \leq M^2(\|f\|_p + M)^{p-2}$ we have that

$$\|f - v_\varepsilon\|_p \leq \|f - v^*\|_p + \beta\|v - v^*\|_0^2 \tag{9}$$

holds for $\beta = M^2(\|f\|_p + M)^{p-2} > 0$ since $|\varepsilon| = \|v - v^*\|_p$.

The proof of (ii) is quite similar. First of all, the work of [1] shows that for each fixed $M > 0$ there is a positive constant $\gamma = \gamma(f, M)$ such that

$$\|f - v\|_p \geq \|f - v^*\|_p + \gamma \|v - v^*\|_p^p \tag{10}$$

for all $v \in V$ satisfying $\|v\|_p \leq M$. Furthermore, setting $v_\varepsilon = v^* + \varepsilon\hat{v}$ where $\|\hat{v}\|_p = 1$ and $\mu\{\text{supp}(f - v^*) \cap \text{supp}(\hat{v})\} = 0$, gives that $F(\varepsilon) = (\int |f - v^*|^p \, d\mu + \int |\varepsilon\hat{v}|^p \, d\mu)^{1/p}$. For $M^{1/p} \geq x \geq 0$ define $G(x) = (\int |f - v^*|^p \, d\mu + x\int |\hat{v}|^p \, d\mu)^{1/p}$. Then for $0 \leq \varepsilon \leq M$, the mean value theorem gives that $G(\varepsilon^p) - G(0) = G'(\eta)|\varepsilon|^p$ for some $\eta, 0 < \eta < \varepsilon^p$. Since $G'(\eta) = (1/p)(\int |f - v^*|^p \, d\mu + \eta\int |\hat{v}|^p \, d\mu)^{1/p - 1}(\int |\hat{v}|^p \, d\mu)$ and $|G'(\eta)| \leq (1/p)\, \|f - v^*\|_p^{1-p}\, \|\hat{v}\|_p^p$, one has that local strong uniqueness of order at best p holds for f at v^* since $|\varepsilon| = \|v_\varepsilon - v^*\|_p$. ∎

We now consider L^p, $1 < p < 2$. Unlike the $2 \leq p < \infty$ case, where precisely two local strong uniqueness orders are possible for a given $f \in L^p \backslash V$, we find here that the local strong uniqueness result is considerably more complicated. The results of [1] show that, depending on f, local strong uniqueness of at worst order 2 or p holds. Also, note that part (ii) of Theorem 1 did not use the assumption that $p \geqslant 2$. Thus we have that

THEOREM 2. *Let* $1 < p < 2, f \in L^p \backslash V$ *with* v^* *the unique best* L^p *approximation from* V *to* f. *Fix* $M > 0$ *and assume that* $\mu\{\text{supp}(f - v^*) \cap \text{supp}(v)\} = 0$ *for each* $v \in V$. *Then local strong uniqueness of order p holds for f at* v^*.

Proof. By [1] we have that in this case local strong uniqueness of order at worst p holds for f at v^*. By the proof of (ii) in Theorem 1 we have that local strong uniqueness of order at best p also holds for f at v^*. ∎

If the supports of $f - v^*$ and V are not disjoint, we have that local strong uniqueness of order r, for any $r \in (p, 2]$ can hold. In this setting, it is also possible that local strong uniqueness holds at a rate which does not correspond to any order. The following examples illustrate this.

EXAMPLE 2. In $L^p[-1, 1]$, for $\alpha > 0$, let $f(x) = |x|^\alpha \text{sgn}(x)$ and let $V = \{g(x)\colon g(x) \equiv c, c \in R\}$. It is clear that $c^* = 0$ is the unique best L^p approximation from V to f for each p, $1 < p < \infty$. It is also easily seen that $\|c - 0\|_p = 2|c|$ for $c \in V$ and $\|f\|_p = (2/(\alpha p + 1))^{1/p}$. For $c \neq 0$, $\|f - c\|_p^p = \int_{-1}^1 | |x|^\alpha \text{sgn}(x) - c|^p \, dx$. Without loss of generality, we may assume that $c > 0$. Then

$$\int_{-1}^{1} | |x|^\alpha \text{sgn}(x) - c|^p \, dx = \int_{-1}^{0} ((-x)^\alpha + c)^p \, dx + \int_0^{c^{1/\alpha}} (c - x^\alpha)^p \, dx + \int_{c^{1/\alpha}}^1 (x^\alpha - c)^p \, dx. \tag{11}$$

Each of these integrals may be evaluated by parts. This yields

$$\int_{-1}^{0} ((-x)^\alpha + c)^p \, dx = \frac{\alpha p}{\alpha p + 1} \left[\left. \frac{x((-\alpha) + c)^p}{\alpha p} \right|_{-1}^{0} + c \int_{-1}^{0} ((-x)^\alpha + c)^{p-1} \, dx \right].$$

Likewise

$$\int_0^{c^{1/\alpha}} (c - x^\alpha)^p \, dx = \left[\frac{\alpha p}{\alpha p + 1} \left. \frac{x(c - x^\alpha)^p}{\alpha p} \right|_0^{c^{1/\alpha}} + c \int_0^{c^{1/\alpha}} (c - x^\alpha)^{p-1} \, dx \right]$$

and

$$\int_{c^{1/\alpha}}^{1} (x^{\alpha}-c)^{p}\,dx=\frac{\alpha p}{\alpha p+1}\left[\frac{x(x^{\alpha}-c)^{p}}{\alpha p}\bigg|_{c^{1/\alpha}}^{1}\right.$$
$$\left.-c\int_{c^{1/\alpha}}^{1}(x^{\alpha}-c)^{p-1}\,dx\right].$$

Evaluating and combining these quantities, we have that

$$\begin{aligned}\|\,|x|^{\alpha}\mathrm{sgn}(x)-c\|_{p}^{p}&=\frac{(1+c)^{p}}{\alpha p+1}\\&\quad+\frac{\alpha pc}{\alpha p+1}\int_{-1}^{0}(c+(-x)^{\alpha})^{p-1}\,dx\\&\quad+\frac{\alpha p}{\alpha p+1}\int_{0}^{c^{1/\alpha}}(c-x^{\alpha})^{p-1}\,dx\\&\quad+\frac{(1-c)^{p}}{\alpha p+1}-\frac{\alpha p}{\alpha p+1}\int_{c^{1/\alpha}}^{1}(x^{\alpha}-c)^{p-1}\,dx\\&=\frac{(1+c)^{p}+(1-c)^{p}}{\alpha p+1}\\&\quad+\frac{\alpha pc}{\alpha p+1}\int_{0}^{c^{1/\alpha}}\{(c+x^{\alpha})^{p-1}+(c-x^{\alpha})^{p-1}\}\,dx\\&\quad+\frac{\alpha pc}{\alpha p+1}\int_{c^{1/\alpha}}^{1}\{(c+x^{\alpha})^{p-1}-(x^{\alpha}-c)^{p-1}\}\,dx.\end{aligned}\tag{12}$$

Note that

$$\frac{(1+c)^{p}+(1-c)^{p}}{\alpha p+1}=\frac{2}{\alpha p+1}+\frac{p(p-1)}{\alpha p+1}c^{2}+O(c^{4})$$

for some ρ with $c<\rho<1$, and the two integrals can be bounded as follows. By the mean value theorem, $\int_{c^{1/\alpha}}^{1}\{(x^{\alpha}+c)^{p-1}-(x^{\alpha}-c)^{p-1}\}\,dx=\int_{c^{1/\alpha}}^{1}(p-1)\,\xi_{x}^{p-2}(2c)\,dx$ for some ξ_{x}, $|\xi_{x}-x^{\alpha}|<c$. To obtain a lower estimate, we replace ξ_{x} by $2x^{\alpha}$ since $p-2<0$ and $x^{\alpha}+c\leqslant 2x^{\alpha}$ on $[c^{1/\alpha},1]$ and get that

$$\int_{c^{1/\alpha}}^{1}\{(x^{\alpha}+c)^{p-1}-(x^{\alpha}-c)^{p-1}\}\,dx\geqslant 2^{p-1}(p-1)c\int_{c^{1/\alpha}}^{1}x^{\alpha(p-2)}\,dx.$$

Depending upon $\alpha(p-2)$ we have the following estimates: For $\alpha(p-2)+1<0$,

$$\int_{c^{1/\alpha}}^{1} \{(x^\alpha+c)^{p-1}-(x^\alpha-c)^{p-1}\}\,dx \geqslant \frac{2^{p-1}(p-1)}{\alpha(2-p)-1}\,(c^{p-1+1/\alpha}-c),$$

where $c^{p-1+1/\alpha}$ dominates for small c since $p-1+1/\alpha<1$. For an upper estimate we have that

$$\begin{aligned}
&\int_{c^{1/\alpha}}^{1} \{(x^\alpha+c)^{p-1}-(x^\alpha-c)^{p-1}\}\,dx \\
&\quad=\int_{c^{1/\alpha}}^{2c^{1/\alpha}} \{(x^\alpha+c)^{p-1}-(x^\alpha-c)^{p-1}\}\,dx \\
&\qquad+\int_{2c^{1/\alpha}}^{1} \{(x^\alpha+c)^{p-1}-(x^\alpha-c)^{p-1}\}\,dx \\
&\quad\leqslant (2^\alpha+1)^{p-1}c^{(p-1+1/\alpha)} \\
&\qquad+\int_{2c^{1/\alpha}}^{1} \{(x^\alpha+c)^{p-1}-(x^\alpha-c)^{p-1}\}\,dx \\
&\quad\leqslant (2^\alpha+1)^{p-1}c^{(p-1+1/\alpha)} \\
&\qquad+2(p-1)\left[\frac{2^\alpha-1}{2^\alpha}\right]^{p-2} c\int_{2c^{1/\alpha}}^{1} x^{\alpha(p-2)}\,dx.
\end{aligned}$$

So that

$$\begin{aligned}
&\int_{c^{1/\alpha}}^{1} \{(x^\alpha+c)^{p-1}-(x^\alpha-c)^{p-1}\}\,dx \\
&\qquad\leqslant\left[(2^\alpha+1)^{p-1}+\frac{4(p-1)(2^\alpha-1)^{p-2}}{\alpha(2-p)-1}\right]c^{(p-1+1/\alpha)}+O(c).
\end{aligned}$$

Likewise, one has (in a straightforward manner) that

$$\begin{aligned}
c^{p-1+1/\alpha} &\leqslant \int_0^{c^{1/\alpha}} \{(x^\alpha+c)^{p-1}+(x^\alpha-c)^{p-1}\}\,dx \\
&\leqslant (2^{p-1}+1)\,c^{p-1+1/\alpha},
\end{aligned}$$

where the estimates are independent of the size of $p-1+1/\alpha$. Thus when $p-2+1/\alpha<0$ we have that for small c there exist positive constants δ and γ such that

$$\left[\frac{2}{\alpha p+1}\right]^{1/p} + \delta c^{p+1/\alpha} \leqslant \| \, |x|^{\alpha}\operatorname{sgn}(x) - c\|_p$$
$$\leqslant \left[\frac{2}{\alpha p+1}\right]^{1/p} + \gamma c^{p+1/\alpha}.$$

Since $\| \, |x|^{\alpha}\operatorname{sgn}(x)\|_p = [2/(\alpha p+1)]^{1/p}$, it follows that local strong uniqueness of order $p+1/\alpha$ holds for $|x|^{\alpha}\operatorname{sgn}(x)$ at $c^* = 0$. That is, one can achieve local strong uniqueness of orders r, $r \in (p, 2)$ for this L^p approximation problem.

Next consider this problem when $P - 2 + 1/\alpha = 0$. Here once again the dominating term for small c is $\int_{c^{1/\alpha}}^{1} \{(x^{\alpha}+c)^{p-1} - (x^{\alpha}-c)^{p-1}\}\, dx$ and proceeding exactly as above one obtains for $0 < c < 1$ that

$$\frac{2^{p-1}(p-1)c}{\alpha} \ln(1/c) \leqslant \int_{c^{1/\alpha}}^{1} \{(x^{\alpha}+c)^{p-1} - (x^{\alpha}-c)^{p-1}\}\, dx$$
$$\leqslant \frac{2(p-1)(2^{\alpha}-1)^{p-1}c}{\alpha 2^{\alpha(p-1)}} \ln(1/c) + O(c).$$

Since the other estimates remain the same one has that there exist positive constants δ and γ such that for small c,

$$\delta c^2 \ln(1/c) \leqslant \| \, |x|^{\alpha} \operatorname{sgn}(x) - c\|_p - \| \, |x|^{p} \operatorname{sgn}(x) - 0\|_p \leqslant \gamma c^2 \ln(1/c).$$

This shows that in this case local strong uniqueness of rate ϕ, $\phi(t) = \gamma t^2 \ln(1/t)$ holds for $|x|^{\alpha} \operatorname{sgn}(x)$ at $c^* = 0$ with respect to L^p. Similarly, for $p - 2 + 1/\alpha > 0$ it can be seen that local strong uniqueness of order 2 holds.

This example shows that local strong uniqueness in L^p for $1 < p < 2$ is of an entirely different character than for $2 < p < \infty$ where only two distinct possibilities exist. Further, it seems likely that it should be possible to find examples where local strong uniqueness of rate ϕ, for any ϕ "between" $\phi_p(t) = t^p$ and $\phi_2(t) = t^2$, holds in L^p, $1 < p < 2$.

As noted earlier, global strong uniqueness results were given by Smarzewski [13]. Specifically for $p \geqslant 2$, the estimate

$$\|f - v\|_p^p \geqslant \|f - v^*\|_p^p + 2^{2-p} \|v - v^*\|_p^p \tag{13}$$

and for $1 < p < 2$,

$$\|f - v\|_p^2 \geqslant \|f - v^*\|_p^2 + c_p \|v - v^*\|_p^2. \tag{14}$$

It should be noted that if the condition $\mu(\operatorname{supp}(f - v^*) \cap \operatorname{supp}(v)\} = 0$ holds, then one has

$$\|f - v\|_p^p \geqslant \|f - v^*\|_p^p + \|v - v^*\|_p^p,$$

whereas, if this condition does not hold the true local behaviour of the best approximation problem at v^* is lost in (13) for $p > 2$. This is true because $\|v - v^*\|_p$ small implies that in (13)

$$\|f - v\|_p \geq \|f - v^*\|_p + \frac{2^{2-p}}{p\,\|f - v^*\|_p^p}\,\|v - v^*\|_p^p + O(\|v - v^*\|_p^{2p}),$$

whereas, the correct local estimate is given by (8). Furthermore, for $p = 2$, one has that v^* is the best approximation to f from V if and only if $(f - v^*)$ is orthogonal to V. Thus by the Pythagorean theorem, one has that $\|f - v\|_2^2 = \|f - v^*\|_2^2 + \|v - v^*\|_2^2$. However, for the case where $1 < p < 2$ and there exists $v \in V$ with $\mu\{\text{supp}(f - v^*) \cap \text{supp}(v)\} \neq 0$, one has that the global estimate gives a local strong uniqueness of at worst 2 estimate for the approximation problem which may be sharp depending upon f.

A final fact that illustrates the utility of a local strong uniqueness approach can be seen in the derivation of corresponding local Lipschitz conditions. Thus, for example, in L^p for $p > 2$ a local Lipschitz condition of order $1/p$ for the best L^p approximation operator using (13) is given in [12, Cor. 4.2]. However, in most standard L^p problems (for example f continuous and V Haar) one will not have disjoint supports occurring for any nonzero $v \in V$, so that the true Lipschitz order of the best approximation operator is $\frac{1}{2}$. For large p, this is a significant improvement.

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